

On relative OR-complexity of Boolean matrices and their complements *

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We construct explicit Boolean square matrices whose rectifier complexity (OR-complexity) differs significantly from the complexity of the complement matrices. This note can be viewed as an addition to the material of [2, §5.6].

Recall that *rectifier* (m, n) -*circuit* is an oriented graph with n vertices labeled as inputs and m vertices labeled as outputs. *Rectifier circuit* (OR-circuit) implements a Boolean $m \times n$ matrix $A = (A[i, j])$ iff for any i and j the value $A[i, j]$ indicates the existence of an oriented path from j -th input to i -th output. Complexity of a circuit is the number of edges in it, circuit depth is the maximal length of an oriented path. See details in [2, 5].

We denote by $\text{OR}(A)$ the complexity of an edge-minimal circuit implementing a given matrix A ; if we speak about circuits of depth $\leq d$, then the corresponding complexity is denoted by $\text{OR}_d(A)$.

It was proved in [2] via method [3] the existence of $n \times n$ -matrices A satisfying

$$\text{OR}(\bar{A})/\text{OR}(A) = \Omega(n/\log^3 n).$$

Note that due to general results [5, 6] on the asymptotic complexity of the class of Boolean matrices the ratio in the question cannot exceed $\Theta(n/\log n)$.

A k -*rectangle* is an all-ones $k \times k$ matrix. A matrix is k -*free* if it does not contain a k -rectangle as a submatrix.

It was established in [2] the existence of an $n \times n$ matrix A simple for depth-2 circuits, $\text{OR}_2(A) = O(n \log^2 n)$, whose complement matrix \bar{A} is 2-free and has relatively high weight (the number of ones) $|\bar{A}| = \Omega(n^{5/4})$. As a consequence of [6], $\text{OR}(\bar{A}) = \text{OR}_2(\bar{A}) = |\bar{A}|$.

Below, we provide an explicit construction of matrices satisfying similar conditions.

Theorem 1. (i) *For an explicit Boolean $n \times n$ matrix C :*

$$\text{OR}(\bar{C})/\text{OR}(C) = n \cdot 2^{-O(\sqrt{\ln n \ln \ln n})}.$$

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(ii) For an explicit Boolean $n \times n$ matrix C the following conditions hold: $\text{OR}(C) = O(n)$, matrix \bar{C} is 2-free and $|\bar{C}| = \Omega(n^{4/3})$.

(Recall that the weight of any 2-free matrix is at most $n^{3/2} + n$.)

The proof of the theorem is based on the following simple combinatorial lemma.

Lemma 1. *Let the weight of an $n \times n$ matrix A be $|A| \geq 2n^{3/2}$. Then A contains $\Omega((|A|/n)^4)$ 2-rectangles.*

Proof. Say that a row *covers* a pair u of two columns, if this row has ones in these columns. If a_i denotes the number of ones in the i -th row of A , then the number of pairs of columns covered by the rows of A is

$$\sigma = \sum_{i=1}^n \binom{a_i}{2} = \frac{1}{2} \sum_{i=1}^n a_i^2 - \frac{|A|}{2} \geq \frac{(\sum_{i=1}^n a_i)^2}{2n} - \frac{|A|}{2} = \frac{|A|^2}{2n} - \frac{|A|}{2} \geq \frac{|A|^2}{4n}.$$

Let b_u be the number of rows covering the pair u of columns. Then $\sum_u b_u = \sigma$. Thus, the number of 2-rectangles in A is

$$\begin{aligned} \sum_u \binom{b_u}{2} &= \frac{1}{2} \sum_u b_u^2 - \frac{\sigma}{2} \geq \frac{(\sum_u b_u)^2}{n(n-1)} - \frac{\sigma}{2} = \\ &= \frac{\sigma^2}{n(n-1)} - \frac{\sigma}{2} \geq \frac{\sigma^2}{2n^2} = \Omega\left(\left(\frac{|A|}{n}\right)^4\right). \end{aligned}$$

□

Let $n = \binom{m}{2}$. Given an $m \times m$ matrix A construct an $n \times n$ matrix B as follows. Label rows and columns of B by 2-element subsets of $[m]$. Set $B[a, b] = 1$ iff $a \times b$ forms a 2-rectangle in A .

Lemma 2. *If A is k -free, then B is K -free, $K = \binom{k-1}{2} + 1$.*

Proof. Suppose that B contains a K -rectangle at the intersection of rows s_1, \dots, s_K and columns t_1, \dots, t_K . Then A contains a rectangle at the intersection of rows $\cup s_i$ and columns $\cup t_i$. But necessarily $|\cup s_i|, |\cup t_i| \geq k$, contradicting k -freeness of A . □

Lemma 3. *If A is k -free and $|A| \geq 2m^{3/2}$, then*

$$\text{OR}(B) = \Omega\left(\left(\frac{|A|}{kn}\right)^4\right),$$

on the other hand, $\text{OR}_3(\bar{B}) = O(n)$.

Proof. By Lemma 1, $|B| = \Omega((|A|/n)^4)$, and Lemma 2 implies that B is K -free. Therefore, by the Nechiporuk's theorem [6]

$$\text{OR}(B) \geq \frac{|B|}{K^2} = \Omega\left(\left(\frac{|A|}{kn}\right)^4\right).$$

We are left to show that the matrix \bar{B} can be implemented by a depth-3 circuit of linear complexity. Take a depth-3 circuit where the nodes on the second and the third layer are numbers $1, \dots, m$, and there is an edge joining an input or an output a with a node i iff $i \in a$. The edges between the second and the third layers are drawn according to the entries of the matrix \bar{A} .

By the construction, the circuit has $O(m^2)$ edges. Indeed, it implements the matrix \bar{B} since there exists a path connecting an input a with an output b iff the submatrix at the intersection of rows b and columns a is not all-zero. \square

To prove p. (i) of the Theorem take $m \times m$ norm-matrix A [4], which is Δ -free and has m^2/Δ ones, where $\Delta = 2^{O(\sqrt{\log m \log \log m})}$, under appropriate choice of parameters. Put $C = \bar{B}$.

To prove p. (ii) take 3-free $m \times m$ Brown's matrix A [1] of weight $\Theta(m^{5/3})$. Put $C = \bar{B}$. \square

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